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A Decomposition Method for Global and Local Quadratic Minimization ⋆

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Abstract. We present a decomposition method for indefinite quadratic programming problems having n variables and m linear constraints. The given problem is decomposed into at most m QP subproblems each having m linear constraints and $n-1$ variables. All global minima, all isolated local minima and some of the non-isolated local minima for the given problem are obtained from those of the lower dimensional subproblems. One way to continue solving the given problem is to apply the decomposition method again to the subproblems and repeatedly doing so until subproblems of dimension 1 are produced and these can be solved directly. A technique to reduce the potentially large number of subproblems is formulated.

Key words: Local and global optimization, Non-convex quadratic program, Parametric linear programming.

1. Introduction

We propose a method to solve

$$\min\{c'x + x'Cx \mid Ax \leqslant b\}, \tag{1.1}$$

where C is (n, n), symmetric and indefinite, c and x are n-vectors, A is an (m, n) matrix and b is an m-vector. Matrix transposition is denoted by a prime ($'$) and all non-primed vectors will be assumed to be column vectors. I_n will denote the (n, n) identity matrix and E^n will denote Euclidean n-space. Let

$$S = \{x \mid Ax \leqslant b\}$$

denote the feasible region for (1.1). When it is necessary to discuss the individual constraints for (1.1), we will write S in the form

$$S = \{x \mid a_i'x \leqslant b_i, i = 1, \ldots, m\},$$

so that a_i' denotes the ith row of A.

We require the following to be satisfied throughout this paper.

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ASSUMPTION 1.1. *(a) $S \neq \emptyset$, and (b) S is bounded.*

The primary difficulty in solving (1.1) is that it may possess many local minimizers. We address this difficulty by utilizing results from our previous paper [4], which we briefly summarize here. In [4], parametric quadratic programming is used to find all isolated local minimizers and some non-isolated local minimizers for the non-convex QP

$$\min\{c'x + x'DQ'x \mid Ax \leqslant b\} \tag{1.2}$$

where c, A, b and x are as in (1.1), and D and Q are (n, k) matrices with $k < n$. The method proceeds by formulating the parametric LP

$$\min\{c'x + t'Q'x \mid Ax \leqslant b,\ D'x = t\}, \tag{1.3}$$

where t is a parameter vector in E^k. Letting $R(t)$ denote the set of feasible solutions for (1.3), the derived problem for (1.2) is

$$\min\{f(t) \mid t \in \mathrm{E}^k\}, \tag{1.4}$$

where

$$f(t) = \begin{cases} \min\{c'x + t'Q'x \mid x \in R(t)\}, & \text{if } R(t) \neq \phi, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.5}$$

It is shown in [4] that the isolated local minimizers of (1.2) and (1.4) are in one to one correspondence. In particular, if t^* is a local minimizer for (1.4) then any optimal solution for (1.3) with $t = t^*$ is a local minimizer for (1.2).

Although the theory was developed for arbitrary $k < n$, the numerical procedures developed in [4] were limited to the case of $k = 1$ in which case (1.3) has just a single parameter and can be solved using the methods in [1]. In this paper, we address the problem of $k = n - 1$ by using a decomposition approach. We begin with the model problem (1.1). We then give a method which will either construct matrices D and Q satisfying $x'DQ'x = x'Cx$ (so that the model problem (1.1) can be rewritten in the model form (1.2)), or, determine that no such matrices D and Q exist.

When such D and Q do indeed exist, the decomposition method then generates m subproblems each of dimension $n - 1$, where m is the number of constraints in (1.1). Combining the local/global solutions of all of these m subproblems gives the local minima and the global minimum for the given problem. Each of these smaller problems is in turn decomposed into m subproblems with their dimension reduced by 1. The process continues by generating smaller and smaller dimensional subproblems until the subproblem can be solved directly. One possibility is a one-dimensional subproblem. Other possibilities will be discussed.

For the case that no such D and Q exist, we will show that either C is positive semi-definite or negative semi-definite. In the former case, (1.1) is a convex quadratic programming problem and may be solved by any convex QP algorithm. In the latter case, (1.1) is a concave quadratic minimization problem. This is a very difficult problem and we have no suggestions as to how it may be solved.

We will give numerical examples to illustrate both the decomposition method and the subproblem reduction procedure.

2. A Decomposition Method

Here we present a decomposition method for (1.1). We will begin with a detailed statement of the method and then establish its properties in Theorem 2.1. We will give an example of applying the method to a small numerical problem. The method requires that C be expressed in a different form. This is performed by Procedure Ψ_1 which is developed in detail in the Appendix.

2.1. DECOMPOSITION METHOD

Use Procedure $\Psi_1(C)$ to obtain matrices Q and D, each having dimensions $(n, n-1)$ and satisfying $C = \frac{1}{2}[DQ' + QD']$ and $\mathrm{rank}(D) = n-1$. Then for $k = 1, 2, \ldots, m$ construct and solve the following subproblems. Define the (n, n) matrix $B_k' = [D, a_k]$. If B_k is singular, the kth subproblem is considered vacuous. If B_k is nonsingular, partition B_k^{-1} as $B_k^{-1} = [H_k, d_k]$, where H_k is $(n, n-1)$ and d_k is the last column of B_k^{-1}. The kth subproblem is then to find all local minimizers for the $n-1$ dimensional problem in t

$$\left.\begin{array}{ll} \text{minimize}: & b_k c' d_k \ + \ (H_k' c \ + \ b_k Q' d_k)' t \ + \ t' H_k' C H_k t \\ \text{subject to}: & a_i' H_k t \ \leqslant \ b_i - b_k a_i' d_k, \quad i = 1, \ldots, m, \ \ i \neq k, \\ \text{and} & d_k' Q t \ \leqslant \ -c' d_k. \end{array}\right\} \tag{2.1}$$

For the kth subproblem, let p_k denote the number of local minimizers so obtained, let these local minimizers be denoted by t_{ki}, $i = 1, \ldots, p_k$ and let $R_k(t)$ denote the feasible region for (2.1). For $i = 1, \ldots, p_k$, $x_{ki} = H_k t_{ki} + b_k d_k$ is a local minimum for (1.1), provided t_{ki} is also a local minimizer for all subproblems j such that $t_{ki} \in R_j$. A global minimizer for (1.1) is then the local minimizer which gives the best objective function value.

In the following theorem, we formulate the properties of the Decomposition Method. The proof of the theorem is constructive and gives insight into the method.

THEOREM 2.1. *Let Assumption 1.1 be satisfied, assume C is indefinite and that the Decomposition Method is applied to (1.1). Then the method determines all global minimizers, all of the isolated local minimizers and some of the non-isolated local minimizers for (1.1).*

Proof. Because C is indefinite, Procedure $\Psi_1(C)$ produces two matrices D and Q, each of dimension $(n, n-1)$ and with $\mathrm{rank}(D) = n-1$. Thus, (1.1) is equivalent to (1.2) with $k = n-1$. The parametric LP corresponding to (1.3) is then

$$\min\{c'x \ + \ t'Q'x \mid D'x \ = \ t, \ \ a_i'x \ \leqslant \ b_i, \ \ i = 1, \ldots, m\}, \tag{2.2}$$

where t has dimension $n-1$. Because of Assumption 1.1(a), the feasible region for (2.2) is non-null for certain t. Furthermore, from Assumption 1.1(b), there is an optimal solution of (2.2) which is an extreme point for all t for which (2.2) has a feasible solution. Because $\text{rank}(D) = n-1$, there are at most m extreme points for (2.2) each corresponding to the $(n-1)$ equality constraints of (2.2) plus one of the m inequality constraints. Consider the kth possibility. Let B_k be as in the statement of the Decomposition Method. If B_k is singular, then the kth inequality constraint does not give rise to an extreme point and the kth subproblem need not be considered further.

Otherwise, the extreme point $x = x(t)$ is the solution of the n simultaneous linear equations

$$D'x(t) = t, \qquad a_k' x(t) = b_k.$$

Letting H_k and d_k be as in the statement of the Decomposition Procedure, it follows that

$$x(t) = H_k t + b_k d_k \tag{2.3}$$

is optimal for (2.2) for all t such that $x(t)$ satisfies primal and dual feasibility for (2.2). Primal feasibility is accounted for by substituting $x(t)$ into the remaining constraints of (2.2). This gives

$$a_i' H_k t \leqslant b_i - b_k a_i' d_k, \quad i = 1, \ldots, m, \; i \neq k. \tag{2.4}$$

Since the first $(n-1)$ constraints of (2.2) are equality constraints, their dual variables are unconstrained in sign. The kth inequality constraint is active at $x(t)$ and its dual variable, namely $-(c+Qt)'d_k$, must be non-negative. This requirement reduces to

$$d_k' Q t \leqslant -c' d_k. \tag{2.5}$$

Observe that the set of all t satisfying (2.4) and (2.5) is precisely $R_k(t)$ in the statement of the Decomposition Method. Thus, we have shown that $x(t)$ given by (2.3) is optimal for (2.2) for all $t \in R_k(t)$.

We next use (2.3) to express the optimal objective function value for (2.2), call it $F_k(t)$, explicitly in terms of t:

$$F_k(t) = c'x + t'Q'x = b_k c' d_k + (H_k' c + b_k Q' d_k)' t + t' Q' H_k t.$$

The quadratic term can be simplified somewhat. First observe that by writing $B_k B_k^{-1} = I_n$ in terms of its partitions, it follows that

$$D' H_k = I_{n-1}. \tag{2.6}$$

Next, because $2C = QD' + DQ'$, it follows from (2.6) that

$$\begin{aligned} 2CH_k &= QD'H_k + DQ'H_k \\ &= Q + D(Q'H_k). \end{aligned}$$

Multiplying on the left by H_k' gives

$$H_k' C H_k = \frac{1}{2}[H_k' Q + Q' H_k],$$

which shows that

$$F_k(t) = b_k c' d_k + (H_k' c + b_k Q' d_k)' t + t' H_k' C H_k t \tag{2.7}$$

The equivalent of (1.5) for (1.1) can now be expressed as

$$f(t) = \begin{cases} F_1(t), & \text{if } t \in R_1(t), \\ F_2(t), & \text{if } t \in R_2(t), \\ \cdots & \cdots \\ F_m(t), & \text{if } t \in R_m(t). \end{cases} \tag{2.8}$$

Note that the problem

$$\min\{F_k(t) \mid t \in R_k(t)\} \tag{2.9}$$

is precisely the kth subproblem of the Decomposition Method. Let t_{ki}, $i = 1, \ldots, p_k$ be the local minimizers for (2.1) as in the statement of the Decomposition Method. Then these are also local minimizers for (2.9). If for all j such that $t_{ki} \in R_j(t)$, t_{ki} is also a local minimizer for

$$\min\{F_j(t) \mid t \in R_j(t)\},$$

then t_{ki} is a local minimizer for the problem

$$\min\{f(t) \mid t \in \mathrm{E}^{n-1}\},$$

where $f(t)$ is defined by (2.8). It now follows from Theorem 2.4 of [4] that $x_{ki} = H_k t_{ki} + b_k d_k$ is a local minimizer for (1.1) and this completes the proof. □

The Decomposition Method can be used to solve (1.1) as follows. Assuming C is indefinite, applying the Decomposition Method to (1.1) produces m subproblems each having m linear inequality constraints and $n - 1$ variables. Each of these subproblems will be either convex, indefinite or concave and the relevant possibility may be determined by invoking Procedure Ψ_1. In the convex case, the subproblem may be solved by using any convex QP algorithm (eg [3]). For the indefinite case, the Decomposition Method may again be applied to produce m further subproblems each having dimension $n - 2$. For the concave case, an appropriate solution method must be utilized. Assuming the indefinite case applies to each subproblem, eventually one dimensional subproblems will be produced and these can be solved directly. The structure of this method is that of a tree. The top of the tree is (1.1). Beneath it are m subproblems each having dimension $n - 1$. Beneath each $n - 1$ dimensional subproblem are m additional subproblems each of dimension $n - 2$. At the bottom of the tree are the one-dimensional subproblems.

An obvious difficulty of this approach is that the number of subproblems will be exponentially large. However, in Section 3 we will show that performing the decomposition in a particular way will result in the number of subproblems being reduced. The matrices D and Q are not uniquely determined and by constructing them in a particular way, the subproblems may be reduced in number to between 1 and m.

Thereom 2.1 states that the Decomposition Method will determine all global minimizers, all isolated local minimizers and *some* of the non-isolated local minimizers for (1.1). That not *all* are necessarily obtained is a consequence of the underlying parametric method being used. See [4] for examples and discussion.

We illustrate the Decomposition Method and the tree of subproblems in

EXAMPLE 2.1.

$$\begin{array}{ll} \text{minimize :} & -x_1 - 2x_2 - x_3 \ + \ x'Cx \\ \text{subject to :} & 0 \ \leqslant \ x_i \ \leqslant \ 1, \quad i = 1, 2, 3, \end{array}$$

where

$$C = \begin{bmatrix} 2 & -0.5 & 4.5 \\ -0.5 & -1 & -1 \\ 4.5 & -1 & 5 \end{bmatrix} = [DQ' + QD']/2,$$

and for simplicity we take

$$D = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

There are six inequality constraints in the problem. Each generates a subproblem with two variables. The objective function (with the constant omitted) and constraints for each subproblem defined by (2.1) are summarized in Table 2.1. As well, the feasible regions for the subproblems are shown together in Figure 2.1.

In this example, each of the six subproblems has exactly five constraints rather than the six one might expect. This is because, for example, when $x_1 = 0$ is active, its gradient is linearly dependent on that of $x_1 \leqslant 1$ and so produces a constraint of the form $0 \leqslant 1$.

Each of the six two-dimensional problems is non-convex and can be decomposed into at most five one-dimensional problems. Each such one-dimensional problem can be viewed as minimizing a piece wise quadratic function over at most five intervals. In each interval, the quadratic objective function may be strictly convex, strictly concave or linear. It is a straightforward matter to determine the local minima for the one-dimensional case. We omit the details of these calculations. However, a typical one-dimensional piecewise function is illustrated in Figure 2.2.

An alternative way to proceed for this example is to note that the Hessian matrices for each of the six two-dimensional subproblems all have exactly one negative eigenvalue. Each of these six subproblems could be solved using the method

Table 2.1. Two-dimensional subproblems for Example 2.1

Active 'x' constraint	Objective function	Constraints
$x_1 = 0$	$\frac{1}{5}(t_1^2 + 4t_1t_2 - 2t_2^2 + 3t_1 - 7t_2)$	$0 \leqslant t_1 + t_2 \leqslant 5$, $0 \leqslant -2t_1 + 3t_2 \leqslant 5$, $9t_1 + 2t_2 \geqslant 8$
$x_1 = 1$	$\frac{1}{5}({t_1}^2 + 4t_1t_2 - 2{t_2}^2 + 12t_1 - 5t_2 - 8)$	$1 \leqslant t_1 + t_2 \leqslant 6$, $-2 \leqslant -2t_1 + 3t_2 \leqslant 3$, $9t_1 + 2t_2 \leqslant 8$
$x_2 = 0$	$\frac{1}{2}(4{t_1}^2 - 3t_1t_2 - 2{t_2}^2 - 2t_1 + 2t_2)$	$0 \leqslant t_2 \leqslant 2$, $0 \leqslant 2t_1 - 3t_2 \leqslant 2$, $9t_1 + 2t_2 \geqslant 8$
$x_2 = 1$	$\frac{1}{2}(4{t_1}^2 - 3t_1t_2 - 2{t_2}^2 + 7t_1 + 4t_2 - 8)$	$1 \leqslant t_2 \leqslant 3$, $-5 \leqslant 2t_1 - 3t_2 \leqslant -3$, $9t_1 + 2t_2 \leqslant 8$
$x_3 = 0$	$2{t_1}^2 + 3t_1t_2 - t_1 - 3t_2$	$0 \leqslant t_1 + t_2 \leqslant 1$, $0 \leqslant t_2 \leqslant 1$, $9t_1 + 2t_2 \leqslant 8$
$x_3 = 1$	$2{t_1}^2 + 3t_1t_2 - 10t_1 - 5t_2 + 8$	$2 \leqslant t_2 \leqslant 3$, $5 \leqslant t_1 + t_2 \leqslant 6$, $9t_1 + 2t_2 \geqslant 8$

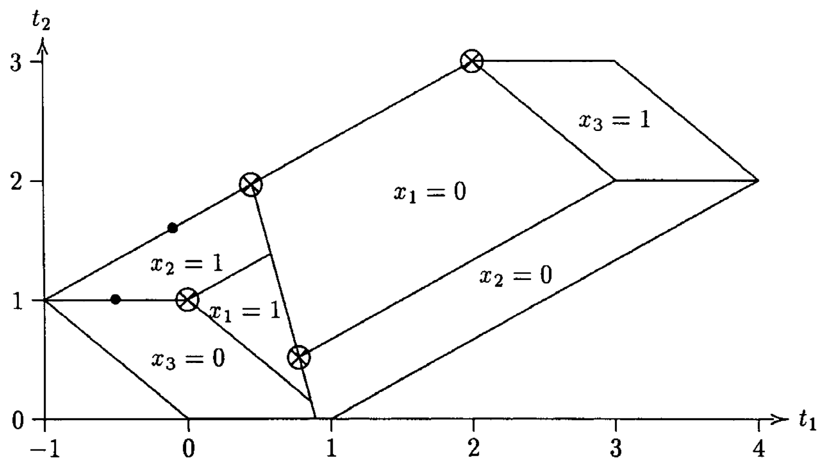

Figure 2.1. Two-dimensional feasible regions for Example 2.1.

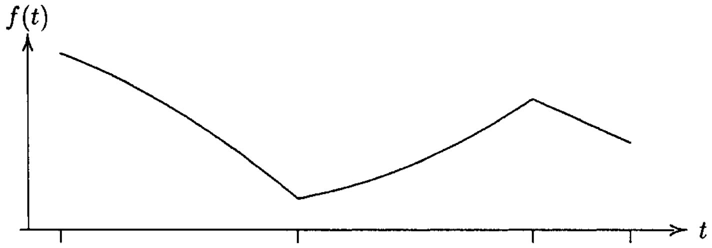

Figure 2.2. Typical piecewise function for one-dimensional subproblem.

Table 2.2. Optimal solutions for Example 2.1 and some variations

Linear term	Optimal type	Objective value	Point in E^2	Point in E^3
$-x_1 - 2x_2 - x_3$	Global	-3.5	$(-0.5, 1)'$	$(0.5, 1, 0)'$
	Local	-3.45	$(-0.1, 1.6)'$	$(0, 1, 0.3)'$
$-x_1 + 2x_2 - x_3$	Global	-0.125	$(0.25, 0)'$	$(0.25, 0, 0)'$
	Local	0.55	$(-0.1, 1.6)'$	$(0, 1, 0.3)'$
	Local	0.5	$(-0.5, 1)'$	$(0.5, 1, 0)'$
$-0.5x_1 + 2x_2 - x_3$	Global	-0.05	$(0.3, 0.2)'$	$(0, 0, 0.1)'$
	Local	0.55	$(-0, 1, 1.6)'$	$(0, 1, 0.3)'$
	Local	-0.03125	$(0.125)'$	$(0.125, 0, 0)'$
	Local	0.71875	$(-0.625, 1)'$	$(0.375, 1, 0)'$

of [4]. For problems with large numbers of constraints, this is a computationally more attractive way to proceed.

Local minima for the two-dimensional problem are indicated in Figure 2.1. The four points indicated by a cross within a circle ($\bigotimes$) are local minima for at least one, but not all the regions in which they lie and thus do not give local minima for the three-dimensional problem. The two points indicated by a bullet ($\bullet$) are local minima for all the regions in which they lie and thus give local minima for the original problem.

The local and global minimizers for Example 2.1 are shown in Table 2.2. Also shown are the local and global solutions for two variations of Example 2.1. These variations are obtained by changing the linear part of the objective function for Example 2.1.

Murty [5] proposes a method to find a global minimum for (1.1). Although his method does not find local minima as does ours, the tree structure of subproblems for his method is analogous to ours. Murty first checks to see if C in (1.1) is positive semi-definite and if so, solves the problem using conventional convex QP algorithms, as do we. Otherwise, for $k = 1, \ldots, m$ the following subproblems are considered:

$$\min\{c'x \; + \; x'Cx \mid a_i'x \leqslant b_i, \; i = 1, \ldots, m, \; i \neq k, \;\; a_k'x \; = \; b_k\}.$$

Each of these m subproblems are considered separately. Each such subproblem includes precisely one equality constraint. This can be used to express one of the variables as an affine function of the remaining variables. Consequently, each subproblem may be reformulated as a problem of the form of (1.1) but with $n - 1$ variables and $m-1$ constraints. Thus, a non-convex problem having n variables and m constraints is decomposed into m subproblems each having $n - 1$ variables and $m - 1$ inequality constraints. Each of these is decomposed into $m - 1$ subproblems each having $n - 2$ variables and $m - 2$ inequality constraints. A particular branch of the method terminates when a convex subproblem is solved or when a one dimensional subproblem is reached. The global minimum for the given problem is obtained by choosing that minimizer for a convex subproblem which gives the smallest objective function value for (1.1).

3. Reduction of the number of subproblems

When the Decomposition Method of Section 2 is applied to (1.1), the method begins by invoking Procedure $\Psi_1(C)$ to construct two $(n, n-1)$ matrices D and Q satisfying

$$C \; = \; \frac{1}{2}(DQ' + QD').$$

Such D and Q are not uniquely determined and it is the purpose of this section to formulate ways to construct D and Q so that the number of subproblems (2.1) is reduced.

Let $\alpha_1, \ldots, \alpha_j$ be a subset of indices of constraints of (1.1) and in addition suppose the gradients of these constraints, namely $a_{\alpha_1}, \ldots, a_{\alpha_j}$, are among the columns of D. Then in the statement of the Decomposition Method, each of the matrices $B_{\alpha_1}, \ldots, B_{\alpha_j}$ will be singular. Consequently subproblems $\alpha_1, \ldots, \alpha_j$ will be omitted. The following result gives a condition under which such D and Q can indeed be constructed.

THEOREM 3.1. *Let M be any (n, n) nonsingular submatrix of A. If $(M^{-1})'CM^{-1}$ contains a (k, k) indefinite principal submatrix, then there exist two $(n, n-1)$ matrices D and Q with* $\operatorname{rank}(D) = n - 1$ *such that*

$$C = \frac{1}{2}(DQ' + QD') \tag{3.1}$$

and at least $n - k$ columns of D are identical to $n - k$ columns of A'. Furthermore, (1.1) can be decomposed into at most $(m - n + k)$ subproblems, each of dimension $n - 1$ by using (3.1).

Proof. Without loss of generality, assume that $M = [a_1, \ldots, a_n]'$ and the (k, k) indefinite principal submatrix E is that induced by the last k rows and columns of $(M^{-1})'CM^{-1}$. Applying Procedure Ψ_1 to E gives $\Psi_1(E) = (D_k, Q_k)$, where the dimensions of both D_k and Q_k are $(k, k-1)$, $\text{rank}(D_k) = k - 1$ and

$$E = \frac{1}{2}(D_k Q_k' + Q_k D_k').$$

Let

$$(M^{-1})'CM^{-1} = \begin{bmatrix} F_1 & F_2 \\ F_2' & E \end{bmatrix},$$

and define

$$\hat{D} = \begin{bmatrix} I_{n-k} & 0 \\ 0 & D_k \end{bmatrix} \quad \text{and} \quad \hat{Q} = \begin{bmatrix} F_1 & 0 \\ 2F_2' & Q_k \end{bmatrix}.$$

This implies

$$(M^{-1})'CM^{-1} = \frac{1}{2}(\hat{D}\hat{Q}' + \hat{Q}\hat{D}'),$$

and thus

$$C = \frac{1}{2}(M'\hat{D}\hat{Q}'M + M'\hat{Q}\hat{D}'M) = \frac{1}{2}(DQ' + QD'),$$

where $D = M'\hat{D}$ and $Q = M'\hat{Q}$. Since $\text{rank}(D_k) = k - 1$, it follows that $\text{rank}(\hat{D}) = n - 1$. Therefore, $\text{rank}(D) = \text{rank}(\hat{D}) = n - 1$ which completes the verification of (3.1). Since $D = M'\hat{D}$, the first $n-k$ columns of D are $a_1, \ldots, a_{n-k}$. As in the statement of the Decomposition Method, the matrices $B_1, \ldots, B_{n-k}$ will each be singular and the $(n-k)$ subproblems $1, \ldots, n-k$ will be omitted. Therefore, (1.1) can be decomposed into at most $(m - n + k)$ subproblems, each of dimension $n-1$, by using the decomposition (3.1). This completes the proof of the theorem. □

The proof of Theorem 3.1 is constructive. For a given matrix M and a specified indefinite principal submatrix E of $(M^{-1})'CM^{-1}$, the proof of Theorem 3.1 constructs matrices D and Q which satisfy the conclusions of Theorem 3.1. In the proof, it is assumed that E corresponds to the last k rows and columns of $(M^{-1})'CM^{-1}$. A detailed procedure to construct D and Q when E stems from arbitrary rows and columns is given in the Appendix (Procedure Ψ_3).

The following example will illustrate both the constructive procedure of the proof of Theorem 3.1 and the conclusions of the theorem.

Example 3.1.
Consider Example 2.1 further. Let $M = I_3$. Then the rows of M are the gradients of

the three upper bound constraints of the problem. Furthermore, $(M^{-1})'CM^{-1} = C$. Note that C contains the indefinite principal submatrix

$$E = \begin{bmatrix} -1 & -1 \\ -1 & 5 \end{bmatrix},$$

which can be decomposed as $E = \frac{1}{2}[D_2Q_2' + Q_2D_2']$ where

$$D_2 = \begin{bmatrix} 1 \\ (\sqrt{6}+1) \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} -1 \\ (\sqrt{6}-1) \end{bmatrix}.$$

This gives

$$D = \hat{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & (\sqrt{6}+1) \end{bmatrix} \quad \text{and} \quad Q = \hat{Q} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \\ 9 & (\sqrt{6}-1) \end{bmatrix}.$$

Applying the Decomposition Method with this D and Q results in the two subproblems corresponding to the constraints $0 \leqslant x_1 \leqslant 1$ being omitted. Although the proof of Theorem 3.1 guarantees only $n-k = 3-2 = 1$ subproblem to be omitted, two are omitted here because their associated gradients are linearly dependent. Two of the remaining four are infeasible. Indeed, for the subproblem generated by the constraint $x_2 = 0$, the constraints are

$$0 \leqslant t_1 \leqslant 1,$$

$$0 \leqslant t_2 \leqslant 1 + \sqrt{6},$$

$$\frac{-1 - 2\sqrt{6} - (10+\sqrt{6})t_1 - 2\sqrt{6}t_2}{\sqrt{6}+1} \geqslant 0,$$

which are clearly infeasible. Similarly, the subproblem generated by $x_3 = 1$ is also infeasible.

For $x_2 = 1$ the subproblem is

$$\begin{array}{ll} \text{minimize :} & -(11+2\sqrt{6})t_1 - (1+2\sqrt{6})t_2 + 2(\sqrt{6}+1)t_1^2 + 9t_1t_2 + (\sqrt{6}-1)t_2^2 \\ \text{subject to :} & 0 \leqslant t_1 \leqslant 1, \quad 1 \leqslant t_2 \leqslant \sqrt{6}+2, \end{array}$$

where the objective function has been multiplied by $(\sqrt{6}+1)$ and the constant term $-(2\sqrt{6}+1)$ has been omitted for ease of presentation.

For $x_3 = 0$ the subproblem is

$$\begin{array}{ll} \text{minimize :} & -t_1 - 2t_2 + 2t_1^2 - t_1t_2 - t_2^2 \\ \text{subject to :} & 0 \leqslant t_1 \leqslant 1, \quad 0 \leqslant t_2 \leqslant 1. \end{array}$$

Table 3.1. Optimal solutions for Example 3.1

Optimal type	Objective value	Point in E^2	Point in E^3
Global	-3.5	$(0, 0.3(\sqrt{6}+1)+1)'$	$(0.5, 1, 0)'$
Local	-3.45	$(0.5, 1)'$	$(0, 1, 0.3)'$

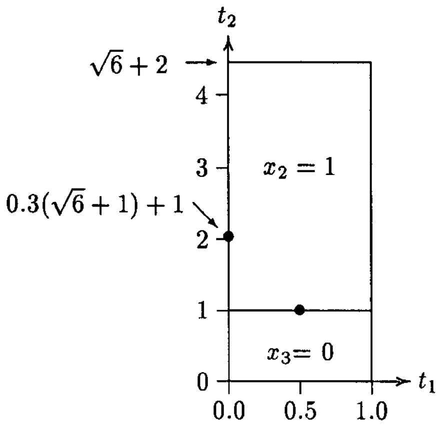

Figure 3.1. Two-dimensional feasible regions for Example 3.1.

The first subproblem has two local minimizers: $(0, 0.3(\sqrt{6}+1)+1)'$ and $(0.5, 1)'$. The second subproblem has just one: $(0.5, 1)'$. Thus $(0.5, 1)'$ and $(0, 0.3(\sqrt{6}+1)+1)'$ are two local minimizers for the two-dimensional problem. The corresponding points in the original space are $(0.5, 1, 0)'$ and $(0, 1, 0.3)'$, respectively, and these are local minima for the given problem, in agreement with Example 2.1. These points are summarized in Table 3.1 and the geometry of the two subproblems is shown in Figure 3.1.

There is considerable flexibility in how Theorem 3.1 may be used. First, an (n, n) submatrix of A must be chosen. Second, a (k, k) indefinite principal submatrix of $(M^{-1})'CM^{-1}$ must be found. According to Theorem 3.1 the maximum subproblem reduction occurs when $k = 2$. One possibility is to enumerate each $(2, 2)$ or $(3, 3)$ principal submatrix and check for indefiniteness. For such small principal submatrices, this will not require much computational effort. However, dealing with much larger principal submatrices could require excessive computation.

The following proposition shows that we can guarantee to omit a certain number of subproblems, but that number can be quite small.

PROPOSITION 3.1. *Assume C has k_1 positive eigenvalues and k_2 negative eigenvalues. Then for any nonsingular (n, n) submatrix M of A, each $(n-\rho, n-\rho)$*

principal submatrix of $(M^{-1})'CM^{-1}$ *will be indefinite or singular, where* $\rho = \min\{k_1, k_2\} - 1$.

Proof. Since C has k_1 positive eigenvalues and k_2 negative eigenvalues, so also does $(M^{-1})'CM^{-1}$. Let

$$(M^{-1})'CM^{-1} = \begin{bmatrix} F_1 & F_2 \\ F_2' & B \end{bmatrix},$$

where B is a $(n-\rho, n-\rho)$ submatrix, F_1 a (ρ, ρ) submatrix and F_2 a $(\rho, n-\rho)$ submatrix. It is sufficient to show that B is indefinite or singular. Suppose to the contrary that B is positive definite or negative definite. Then B is invertible. Hence

$$\begin{bmatrix} I & -F_2B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} F_1 & F_2 \\ F_2' & B \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^{-1}F_2' & B^{-1} \end{bmatrix} = \begin{bmatrix} F_1 - F_2B^{-1}F_2' & 0 \\ 0 & B^{-1} \end{bmatrix}.$$

Assume first that B is positive definite. This implies $(n - \rho) \leqslant k_1$ and thus $\rho \geqslant n - k_1 = k_2$. This contradicts the definition of ρ. Consequently, B is not positive definite. Similarly, the assumption that B is negative definite leads to the contradiction that $\rho \geqslant k_1$. Consequently, B is either indefinite or singular. □

4. Conclusions

Given an indefinite QP with n variables, m linear constraints and a bounded feasible region, we have formulated a Decomposition Method which produces at most m QP subproblems, each of dimension $n-1$. The global minimum, all isolated local minima and some of the non-isolated local minima for the given problem can be obtained from the local minima of the smaller dimensional problems. Each of the $(n-1)$–dimensional problems can be decomposed in a similar manner and so on, until one-dimensional problems are obtained and these can be solved directly.

The method of generating the lower dimensional subproblems is not unique and we have presented a method which can be used to reduce the number of subproblems.

The Decomposition Method and the technique for reducing subproblems are illustrated with small numerical examples.

Appendices

Here we present the technical details of Procedure Ψ_1 required by the Decomposition Method, Procedure Ψ_2 (a conjugate direction procedure) which is used by Procedure Ψ_1, and Procedure Ψ_3 which is used in reducing the number of subproblems for the Decomposition Method.

Appendix A. Procedure Ψ_1 (C)

Given an (n, n) symmetric matrix C, Procedure $\Psi_1(C)$ determines whether or not C is indefinite, positive semi-definite or negative semi-definite. If C is indefinite, Procedure $\Psi_1(C)$ constructs two $(n, n-1)$ matrices D and Q with $\text{rank}(D) = n-1$ and $\text{rank}(Q) = \text{rank}(C) - 1$, and which satisfy $C = \frac{1}{2}[DQ' + QD']$. In this case, we write $\Psi_1(C) = (Q, D)$. The details of Procedure $\Psi_1(C)$ are as follows.

We first require an (n, n) non-singular matrix M and an (n, n) diagonal matrix Λ satisfying $M'CM = \Lambda$, where the diagonal elements of Λ are all either -1, 0 or $+1$. Such matrices may be found by either performing an eigenvalue decomposition for C or by using a modified conjugate direction method described in Subsection Appendix B. The latter method is important because it requires only $O(n^3)$ arithmetic operations. It is straightforward to show that the diagonal elements of Λ are all nonnegative if and only if C is positive semi-definite, and are all non-positive if and only if C is negative semi-definite. In either of these cases, Procedure $\Psi_1(C)$ terminates with the relevant information. The remaining possibility is that Λ has two nonzero diagonal elements of opposite sign and this is equivalent to C being indefinite. In this case, Procedure $\Psi_1(C)$ continues as follows.

Suppose k and l are such that λ_k and λ_l are both nonzero, have opposite signs, and assume $k < l$. Then $\lambda_k + \lambda_l = 0$. Let e_i denote the i-th unit vector of dimension $n - 1$. If $l < n$, define $\hat{D}$ and $\hat{Q}$ according to

$$\hat{D}' = [\, e_1, \ldots, e_{l-1}, e_k, e_l, e_{l+1}, \ldots, e_{n-1} \,], \quad \text{and} \quad \hat{Q}' = \hat{D}'\Lambda.$$

If $l = n$, define

$$\hat{D}' = [e_1, \ldots, e_{l-1}, e_k], \quad \text{and} \quad \hat{Q}' = \hat{D}'\Lambda.$$

Note that $\hat{D}'$ differs from the $(n-1, n-1)$ identity matrix by the insertion of e_k after column $l - 1$. It is straightforward to show that $\hat{D}\hat{Q}'$ differs from Λ only in the (k, l)th and (l, k)th elements which are

$$(\hat{D}\hat{Q}')_{kl} = \lambda_l \quad \text{and} \quad (\hat{D}\hat{Q}')_{lk} = \lambda_k.$$

But then $\lambda_k + \lambda_l = 0$ implies

$$\Lambda = \frac{1}{2}[\hat{D}\hat{Q}' + \hat{Q}\hat{D}'].$$

Thus

$$C = (M')^{-1}\Lambda M^{-1} = \frac{1}{2}[(M')^{-1}\hat{D}\hat{Q}'M^{-1} + (M')^{-1}\hat{Q}\hat{D}'M^{-1}]$$

and thus

$$D = (M')^{-1}\hat{D} \quad \text{and} \quad Q = (M')^{-1}\hat{Q}$$

satisfy the conditions of Procedure Ψ_1. The procedure is then complete with $\Psi_1(C) = (Q, D)$.

Appendix B. Procedure Ψ_2 (*C*): a conjugate direction procedure

Conjugate directions are used in quadratic and nonlinear programming, usually in the context of a positive definite Hessian. The basic ideas are as follows. Let C and M be (n, n) matrices with C being symmetric and positive definite. The columns of M are (normalized) conjugate directions if $M'CM = I_n$. There are many ways to construct conjugate directions. We shall generalize the method developed in [3] for the case of indefinite C. The resulting matrix M will have the property that $M'CM$ is diagonal and the diagonal entries are either $1, 0$ or -1.

Let k be such that $1 \leqslant k \leqslant n$. Let

$$D' = [Cc_1, Cc_2, \ldots, Cc_{k-1}, d_k, \ldots, d_n] \tag{B.1}$$

and suppose

$$D^{-1} = [\beta_1 c_1, \beta_2 c_2, \ldots, \beta_{k-1} c_{k-1}, c_k, \ldots, c_n], \tag{B.2}$$

where $\beta_i = \text{sign}(c_i' C c_i)$ for $i = 1, \ldots, k-1$ and for any scalar θ

$$\text{sign}(\theta) = \begin{cases} 1, & \text{if } \theta \geqslant 0, \\ -1, & \text{if } \theta < 0. \end{cases}$$

The process begins with $k = 1$ and D is any non-singular matrix (e.g., the identity matrix). By definition of the inverse matrix, for all i, j with $1 \leqslant i, j \leqslant k-1$

$$c_i' C c_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j \text{ and sign}(c_i' C c_j) = 1, \\ -1, & \text{if } i = j \text{ and sign}(c_i' C c_j) = -1. \end{cases} \tag{B.3}$$

Defining $M = [c_1, c_2 \ldots, c_n]$ it follows from (B.3) that $M'CM$ has a $(k-1, k-1)$ block diagonal submatrix in the top left corner whose diagonal entries are either 1 or -1.

We next show an updating procedure which will produce a new D and M having the same structure but with $k-1$ replaced by k. There are three cases to be considered.

Case 1. $c_k' C c_k \neq 0$.

For ease of notation, let $\gamma_k = (|c_k' C c_k|)^{-\frac{1}{2}}$. Suppose we obtain a new matrix $\hat{D}'$ from D' by replacing d_k with

$$d = \gamma_k C c_k. \tag{B.4}$$

Letting

$$\hat{D}^{-1} = [\hat{c}_1, \ldots, \hat{c}_n], \tag{B.5}$$

it follows from the Sherman–Morrison formula [6], (see also [2]) that

$$\left.\begin{aligned} \hat{c}_i &= \beta_i c_i - \tfrac{d' c_i}{d' c_k} c_k, \quad && i = 1, \ldots, k-1, \\ \hat{c}_k &= \tfrac{1}{d' c_k} c_k, && \\ \hat{c}_i &= c_i - \tfrac{d' c_i}{d' c_k} c_k, && i = k+1, \ldots, n. \end{aligned}\right\} \tag{B.6}$$

From (B.1), (B.2) and the definition of the inverse matrix we have

$$d'c_i = \gamma_k c_k' C c_i = 0, \quad i = 1, \ldots, k-1. \tag{B.7}$$

Thus, from (B.6)

$$\hat{c}_i = c_i, \quad i = 1, \ldots, k-1, \tag{B.8}$$

i.e., the first $k-1$ columns of $\hat{D}^{-1}$ are unchanged by the update. Furthermore, from (B.6) and (B.4)

$$\hat{c}_k = \frac{\gamma_k^{-1}}{c_k' C c_k} c_k = \operatorname{sign}(c_k' C c_k)\gamma_k c_k = \beta_k(\gamma_k c_k), \tag{B.9}$$

where $\beta_k = \operatorname{sign}(c_k' C c_k)$ Summarizing (B.4), (B.5), (B.7) and (B.9) we have shown

$$\begin{aligned} \hat{D}' &= [Cc_1, \ldots, Cc_{k-1}, C(\gamma_k c_k), d_{k+1}, \ldots, d_n] \\ \hat{D}^{-1} &= [\beta_1 c_1, \ldots, \beta_{k-1} c_{k-1}, \beta_k(\gamma_k c_k), \hat{c}_{k+1}, \ldots, \hat{c}_n]. \end{aligned}$$

where $\beta_k = \operatorname{sign}(c_k' C c_k)$. Thus $\hat{D}'$ and $\hat{D}^{-1}$ are of precisely the same form as D' and D^{-1}, respectively, only with $k-1$ replaced with k.

Case 2. $c_k' C c_k = 0$ and $Cc_k = 0$.
In this case, c_k is in the null space of C and no update need be performed. Furthermore, it follows from (B.6) that c_k will remain unchanged by further updates corresponding to Case 1.

Case 3. $c_k' C c_k = 0$ and $Cc_k \neq 0$.
First, we argue that there is a $j > k$ with

$$c_k' C c_j \neq 0. \tag{B.10}$$

For if not, by definition of the inverse matrix, $c_k' C c_i = 0$ for $i = 1, \ldots, k-1$, $c_k' C c_k = 0$ by the assumption of Case 3 and $c_k' C c_i = 0$ for $i = k+1, \ldots, n$ as in the present assumption. Thus Cc_k is orthogonal to n linearly independent vectors and must be the zero vector. But this is in contradiction to the assumption of Case 3. Consequently, there is a $j > k-1$ satisfying (B.10). Suppose next that we modify D' and D^{-1} by replacing c_k with $c_k + c_j$ and d_j with $d_j - d_k$. Then the modified D' and D^{-1} still satisfy (B.1) and (B.2). Furthermore,

$$(c_k + c_j)' C (c_k + c_j) = 2c_j' C c_k \neq 0$$

and now Case 1 applies to the modified matrices.

Note that D^{-1} is not required for any computations and is only used for expository reasons. We next give a detailed statement of the algorithm.

Procedure Ψ_2 (C)
Start with $D^{-1} = I_n = [c_1, \ldots, c_n]$ and for $k = 1, \ldots, n$, do the following.
Let $D^{-1} = [\beta_1 c_1, \beta_2 c_2, \ldots, \beta_{k-1} c_{k-1}, c_k, \ldots, c_n]$.

Step 1. If $c_k' C c_k \neq 0$, set $\gamma_k = (|c_k' C c_k|)^{-\frac{1}{2}}$, $\beta_k = \operatorname{sign}(c_k' C c_k)$, and $d = \gamma_k C c_k$ and compute $\hat{D}^{-1} = [\hat{c}_1, \ldots, \hat{c}_n]$ where

$$\begin{aligned} \hat{c}_i &= c_i, & i &= 1, \ldots, k-1 \\ \hat{c}_k &= \beta_k \gamma_k c_k, \\ \hat{c}_i &= c_i - \tfrac{d' c_i}{d' c_k} c_k, & i &= k+1, \ldots, n. \end{aligned}$$

Replace D^{-1} with $\hat{D}^{-1}$ and continue.

Step 2. If $c_k' C c_k = 0$ and $C c_k = 0$, set $\beta_k = 1$, leave D^{-1} unchanged and continue.

Step 3. If $c_k' C c_k = 0$, and $C c_k \neq 0$, determine $j > k$ such that $c_k' C c_j \neq 0$, modify D^{-1} by replacing c_k with $c_k + c_j$ and return to Step (1) with the modified D^{-1}.

The properties of the conjugate direction method are summarized in

PROPOSITION B.1. *Let the conjugate direction method be applied to the symmetric matrix C and let $D^{-1} = [\beta_1 c_1, \ldots, \beta_n c_n]$ be the final matrix so obtained. Let $M = [c_1, \ldots, c_n]$. Then $M' C M = \Lambda$, where Λ is a diagonal matrix with all diagonal entries being either -1, 0 or $+1$. The number of arithmetic operations required by it is $O(n^3)$.*

EXAMPLE 4.1. We illustrate Procedure Ψ_2 by applying it to

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In Step 1, $c_1' C c_1 = 0$ and $C c_1 = (0, 1)' \neq 0$ so we go to Step 3 to determine $j = 2$ with $c_1' C c_2 = 1$ and replace $D^{-1} = I_2$ with

$$D^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We now return to Step 1 and determine $c_1' C c_1 = 2$, $\gamma_1 = 1/\sqrt{2}$, $d = 1/\sqrt{2}(1, 1)'$, $\beta_1 = 1$, and,

$$D^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$

Returning to Step 1 for $k = 2$ we compute $c_2' C c_2 = -\frac{1}{2}$, $\gamma_2 = \sqrt{2}$, $d = 1/\sqrt{2}(1, -1)'$, $\beta_2 = -1$ and

$$D^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

From this we obtain

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

and compute $M'CM = \text{diag}(1, -1)$.

Appendix C. Procedure Ψ_3 (*C, M, J, k*)

Theorem 3.1 provides a constructive procedure to obtain matrices D and Q satisfying (3.1) for the case that $(M^{-1})'CM^{-1}$ contains a (k, k) indefinite principal submatrix induced by the last k rows and columns of $(M^{-1})'CM^{-1}$. The purpose of this section is to generalize the construction to the case where $(M^{-1})'CM^{-1}$ has an indefinite principal submatrix in a general position. We refer to this as Procedure Ψ_3.

Procedure Ψ_3 (*C, M, J, k*)
Let C be an (n, n) symmetric matrix and M be an (n, n) nonsingular matrix such that $(M^{-1})'CM^{-1}$ contains a (k, k) indefinite principal submatrix E. Let $J = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be an ordered index set specifying E; i.e.,

$$(E)_{ij} \;=\; ((M^{-1})'CM^{-1})_{\gamma_i \gamma_j}, \quad 1 \leqslant i, j \leqslant k,$$

and assume $\gamma_1 < \gamma_2 < \dots < \gamma_k$. Given the input data C, M, J and k, Procedure $\Psi_3(C, M, J, k)$ constructs two matrices D and Q such that $C = \frac{1}{2}[DQ' + QD']$, $\text{rank}(D) = n - 1$ and at least $n - k$ columns of D are identical to $n - k$ columns of M'. The details of Procedure $\Psi_3(C, M, J, k)$ are as follows.

By invoking Procedure $\Psi_1(E)$, we obtain two $(k, k - 1)$ matrices D_k and Q_k such that

$$E = \frac{1}{2}(D_k Q_k' + Q_k D_k').$$

Let $D_k = (\bar{d}_{\mu\nu})$, $Q_k = (\bar{q}_{\mu\nu})$, $K = \{1, \dots, k\}$ and $(M^{-1})'CM^{-1} = (f_{ij})$. We next construct matrices $\hat{D} = (\hat{d}_{ij})$ and $\hat{Q} = (\hat{q}_{ij})$ according to

$$\hat{d}_{ij} = \begin{cases} 1 & \text{if } i = j,\ i \notin J,\ i < \gamma_k, \\ 1 & \text{if } i = j + 1,\ i > \gamma_k, \\ \bar{d}_{\mu\nu} & \text{if } j < \gamma_k,\ i = \gamma_\mu,\ j = \gamma_\nu \text{ for some } \mu, \nu \in K, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{q}_{ij} = \begin{cases} f_{ij} & \text{if } i, j \notin J, \; j < \gamma_k, \\ 2f_{ij} & \text{if } j \notin J, \; i \in J, \; j < \gamma_k, \\ 2f_{i(j+1)} & \text{if } i \in J, \; j \geqslant \gamma_k, \\ f_{i(j+1)} & \text{if } i \notin J, \; j \geqslant \gamma_k, \\ \bar{q}_{\mu\nu} & \text{if } j < \gamma_k, \; i = \gamma_\mu, \; j = \gamma_\mu \text{ for some } \mu, \nu \in K, \\ 0 & \text{otherwise}. \end{cases}$$

Setting $D = M'\hat{D}$ and $Q = M'\hat{Q}$ completes the construction.

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